THE NULL-CONE AND COHOMOLOGY OF VECTOR BUNDLES ON FLAG MANIFOLDS

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1. Introduction

In this paper we study the null-cone of a semi-simple algebraic group acting on a number of copies of its Lie algebra via the diagonal adjoint action. Such actions and their generalizations have been considered by various authors, see, for example, [KW1, KW2, CM]. Our interest in the questions was motivated by applications in the study of ordinary deformations of Galois representations. This point of view is due to Snowden who studied the case $\mathfrak{g} = \mathfrak{sl}_2$ in [S]. In the case of $\mathfrak{g} = \mathfrak{sl}_2$ he shows that the null-cones (and other related spaces that come up in ordinary deformations of Galois representations), are Cohen-Macaulay but not Gorenstein. In the case of characteristic zero the method of Snowden amounts to proving that the null-cone has rational singularities. In this paper we show that for $\mathfrak{g} = \mathfrak{sl}_3$ the null-cones do still have rational singularities, and hence are Cohen-Macauley. However, we also show that this fails in general. For example, it is not difficult to see that when the group is of type B_2 then the null cone is not normal. In the type A_5 we further show that the normalization of the null-cone does not have rational singularities. We do this by giving estimates on cohomology groups of homogenous vector bundles on flag manifolds. This still leaves open the possibility that the normalizations of the null-cones might be Cohen-Macauley, although that appears unlikely.

The paper is organized as follows. In section 2 we introduce the null-cone and in section 3 we make some general remarks on cohomology of homogenous vector bundles on flag manifolds. In section 4 we reduce the question of rational singularities to a question of cohomology of equivariant vector bundles on flag manifolds. In section 5 we compute large enough piece of the cohomology of certain homogenous vector bundles and in section 6 we state and prove our main results.

As our results in this paper are mostly counterexamples we will just work in characteristic zero and will only comment on the finite characteristic case.

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2. The Null-cone

We work over a field of characteristic zero which we can and will, for simplicity, take to be $\mathbb{C}$. Let $\mathfrak{g}$ be a semi-simple Lie algebra, and let us write G for the corresponding adjoint group $\mathrm{Aut}(\mathfrak{g})^0$.

We write $\mathfrak{b}$ for Borel subalgebras of $\mathfrak{g}$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ for their nil-radicals. Similarly, we write B for Borel subgroups of G.

The group G acts diagonally on $\mathfrak{g}^{\oplus r}$ via the adjoint action. We write

$$\mathcal{A}_r = \{(x_1, \ldots, x_r) \in \mathfrak{g}^{\oplus r} \mid f(x_1, \ldots, x_r) = 0 \ \forall \ f \in \mathbb{C}[\mathfrak{g}^{\oplus r}]_+^G\}$$

for the (reduced) invariant theory null-cone associated to this action. The null-cone itself is, in general, a non-reduced scheme, but in what follows we will work with its underlying reduced scheme structure. It is not difficult to see that this variety can also be described in the following manner (see [KW2]):

$$\mathcal{A}_r = \{(x_1, \ldots, x_r) \in \mathfrak{g}^{\oplus r} \mid \exists \ \text{Borel subalgebra } \mathfrak{b} \text{ such that } x_i \in \mathfrak{n}\}.$$

Furthermore, in the case $\mathfrak{g} = \mathfrak{sl}_n$, we can view $\mathcal{A}_r$ as

$$\mathcal{A}_r = \{(x_1, \ldots, x_r) \in \mathfrak{g}^{\oplus r} \mid x_{i_1} \cdots x_{i_n} = 0 \text{ for all choices of } i_1, \ldots, i_n\}.$$

The variety $\mathcal{A}_r$ has a natural G-equivariant resolution of singularities which can be described as follows. Let X denote the flag variety of G then the resolution is given by:

$$\widetilde{\mathcal{A}}_r := G \times_B \mathfrak{n}^{\oplus r} \cong (T^*X)^{\oplus r} \xrightarrow{\ \varphi_r\ } \mathcal{A}_r$$

where the map φ_r is given by $\varphi_r(g, x_1, \ldots, x_r) = (g\, x_1, \ldots, g\, x_r)$. We write $\pi : \widetilde{\mathcal{A}}_r := G \times_B \mathfrak{n}^{\oplus r} \cong (T^*X)^{\oplus r} \to X$ for the projection.

In the case of $r = 1$ the null-cone $\mathcal{A}_1 = \mathcal{N}$ is the usual nilpotent cone and the resolution $\varphi_1 : T^*X \to \mathcal{N}$ is often referred to as the Springer resolution. In this case the null-cone is reduced and the ring of invariants exhibits it as a complete intersection.

In the case of $r = 2$ Charbonnel and Moreau [CM] defined a nilpotent bicone and they show that it is a complete intersection and is in general not reduced. The variety $\mathcal{A}_2$ is an irreducible component of the bicone.

The case of $\mathfrak{g} = \mathfrak{sl}_2$ was studied by Snowden where he proves that in this case $\mathcal{A}_r$ has rational singularities and so is, in particular, Cohen Macauley. As $\varphi_r : \widetilde{\mathcal{A}}_r \to \mathcal{A}_r$ is a resolution of singularities of an affine variety we conclude that

$$\Gamma(\widetilde{\mathcal{A}}_r, \mathcal{O}_{\widetilde{\mathcal{A}}_r}) \quad \text{is the normalization of} \quad \mathbb{C}[\mathcal{A}_r].$$

It is easy to see that $\mathcal{A}_r$ has rational singularities, i.e., that $R\varphi_{r*}\mathcal{O}_{\widetilde{\mathcal{A}}_r} \cong \mathcal{O}_{\mathcal{A}_r}$ if and only if

$$\mathrm{H}^i(\widetilde{\mathcal{A}}_r, \mathcal{O}_{\widetilde{\mathcal{A}}_r}) = 0 \ \text{ for } i > 0 \qquad \text{and} \qquad \Gamma(\widetilde{\mathcal{A}}_r, \mathcal{O}_{\widetilde{\mathcal{A}}_r}) = \mathbb{C}[\mathcal{A}_r] \, .$$

3. Cohomology of equivariant vector bundles

Let us consider the flag manifold $X = G/B$, where we think of having chosen a particular Borel subgroup B as a base point. Then we have an equivalence of categories

$$\{G\text{-equivariant vector bundles on } X\} \longleftrightarrow \{B\text{-representations}\} \, .$$

Given a B-representation E we will use the same symbol E to denote the corresponding vector bundle. Thus, $\mathrm{H}^k(X, E)$ stands for the cohomology of the G-equivariant vector bundle associated to E. Let us write $T \subset B$ for a maximal torus and $X^*(T)$ for the group of characters $\lambda : T \to \mathbb{G}_m$. We also write $\Phi \subset X^*(T)$ for the roots of T and we choose a positive root system Φ^+ such that the roots in B are negative. We also write

$$X^+(T) \ = \ \{\lambda \in X^*(T) \mid \langle \lambda, \check{\alpha} \rangle \geq 0 \text{ for } \alpha \in \Phi^+\}$$

for the dominant weights; here $\check{\alpha}$ stands for the coroot associated to α. We write $L(\lambda)$ for the irreducible representation of G associated to the highest weight $\lambda \in X^+(T)$. Given a representation V of G we write

$$\mathrm{Supp}(V) = \{\lambda \in X^+(T) \mid L(\lambda) \text{ occurs as a direct summand in } V\}$$

for the support of V; it is, of course, a subset of the dominant weights $X^+(T)$.

We can view each $\lambda \in X^*(T)$ also as a character of B and in this manner λ gives rise to a G-equivariant line bundle $\mathcal{L}_\lambda$. If λ is dominant then $L(\lambda) = \mathrm{H}^0(X, \mathcal{L}_\lambda)$. The "dot" action of the Weyl group W on $X^*(T)$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$; here ρ, as usual, is half the sum of positive roots. We recall the statement of the Borel-Weil-Bott theorem:

If there exists a $w \in W$ such that $w \cdot \lambda \in X^+(T)$ then

(3.1)
$$\mathrm{H}^k(X, \mathcal{L}_\lambda) \simeq \begin{cases} \mathrm{H}^0(X, \mathcal{L}_{w \cdot \lambda}) & \text{if } k = l(w) \\ 0 & \text{otherwise} \, . \end{cases}$$

This statement says, in particular, that if there is no $w \in W$ such that $w \cdot \lambda$ is dominant then the cohomology of $\mathcal{L}_\lambda$ vanishes in all degrees.

Consider a B-representation E and the cohomology groups $\mathrm{H}^k(X, E)$ of the corresponding G-equivariant vector bundle. The cohomology groups $\mathrm{H}^k(X, E)$ are G-representations, of course. We will give a simple upper bound for $\mathrm{Supp}(\mathrm{H}^k(X, E))$. Let us choose a filtration of the B-representation E such that the associated graded $\mathrm{gr}E$ is a direct sum of one dimensional representations, i.e., of characters of T. Let us write $\chi(E)$ for the set of characters appearing in this direct sum decomposition:

$$\mathrm{gr}E \ \cong \ \bigoplus_{\lambda_i \in \chi(E)} \lambda_i^{\oplus n_i} \, .$$

Thus we obtain a filtration of the G-bundle E such that the associated graded is a direct sum of G-equivariant line bundles. From this it is easy to conclude:

$$\text{Supp}(\text{H}^k(X, E)) \subset \{\lambda \in X^+(T) \mid w \cdot \lambda \in \chi(E) \text{ for a } w \in W \text{ with } \ell(w) = k\}.$$

Let us call the right-hand side of this equality the *potential support* of $\text{H}^k(X, E)$ and we write $\text{PSupp}(\text{H}^k(X, E))$ for it.

Let us recall that either by a direct calculation or using the Hodge decomposition for the flag manifold and, identifying $\mathfrak{n}$ and $(\mathfrak{g}/\mathfrak{b})^*$ via the Killing form, we conclude:

$$(3.2) \qquad \text{H}^i(X, \wedge^k \mathfrak{n}) = \begin{cases} 0 \text{ if } i \neq k \\ \text{trivial } G\text{-module of dimension } |\{w \in W \mid l(w) = k\}| \text{ if } i = k. \end{cases}$$

Finally, as part of the direct calculation one makes use of the following:

Lemma 3.3. *If χ is a sum of distinct negative roots and $w \cdot \chi$ is dominant then $w \cdot \chi = 0$.*

For a proof of this lemma see, for example, [J, 6.18 Proposition].

4. SOME REDUCTIONS

Recall that we have reduced the question of normality and rational singularities to the study of

$$\text{H}^i(\widetilde{\mathcal{A}}_r, \mathcal{O}_{\widetilde{\mathcal{A}}_r}) = \text{H}^i((T^*X)^{\oplus r}, \mathcal{O}_{(T^*X)^{\oplus r}}).$$

We have

$$\text{H}^i(\widetilde{\mathcal{A}}_r, \mathcal{O}_{\widetilde{\mathcal{A}}_r}) = \text{H}^i((T^*X)^{\oplus r}, \mathcal{O}_{(T^*X)^{\oplus r}}) = \text{H}^i(X, \text{Sym}((\mathfrak{g}/\mathfrak{b})^{\oplus r})).$$

In the latter equality we use the fact that the tangent bundle of X is $TX = \mathfrak{g}/\mathfrak{b}$ as G-equivariant vector bundles; recall our convention the B-representation $\mathfrak{g}/\mathfrak{b}$ also stands for the corresponding G-equivariant bundle. Thus we are reduced to analyzing the cohomology groups $\text{H}^i(X, \text{Sym}((\mathfrak{g}/\mathfrak{b})^{\oplus r}))$. In particular, $\mathcal{A}_r$ is normal if and only if

$$\mathbb{C}[\mathcal{A}_r] = \text{H}^0(X, \text{Sym}((\mathfrak{g}/\mathfrak{b})^{\oplus r})).$$

We then conclude that

$$\mathcal{A}_r \text{ is normal if and only if the map } \text{Sym}(\mathfrak{g}^{\oplus r}) \to \text{H}^0(X, \text{Sym}((\mathfrak{g}/\mathfrak{b})^{\oplus r})) \text{ is onto}.$$

and

$$(4.1) \qquad \begin{aligned} &\mathcal{A}_r \text{ has rational singularities if and only if it is} \\ &\text{normal and } \text{H}^i(X, \text{Sym}((\mathfrak{g}/\mathfrak{b})^{\oplus r})) = 0 \text{ for } i > 0 \ . \end{aligned}$$

We will next make some very general reductions for the vanishing of the higher cohomology in (4.1). As we will show later, the higher cohomology does not vanish in general and hence the general reductions are not so useful. We will be able to obtain more precise statements later. However, the general remarks below are perhaps helpful as a general guide.

We begin with some simple lemmas:

Lemma 4.2. *Suppose that E is a B-module. Then for all $i \geq 1$,*

$$\mathrm{H}^i(X, \mathrm{Sym}(E^{\oplus r})) = 0 \text{ for all } r \geq 1 \Longleftrightarrow \mathrm{H}^i(X, E^{\otimes r}) = 0 \text{ for all } r \geq 1.$$

Proof. We have $\mathrm{Sym}(E^{\oplus r}) = (\mathrm{Sym}(E))^{\otimes r} = \bigoplus(S^{i_1}E \otimes \cdots \otimes S^{i_r}E)$. So $E^{\otimes r}$ is a direct summand of $\mathrm{Sym}(E^{\oplus r})$ and thus "$\Rightarrow$" follows. In the other direction, assume that $\mathrm{H}^i(X, E^{\otimes r}) = 0$ for all $r \geq 1$. Then we have $\mathrm{H}^i(X, (E^{\oplus r})^{\otimes k}) = 0$ for all $r, k \geq 1$. Now $\mathrm{Sym}^k(E^{\oplus r})$ is a direct summand of $(E^{\oplus r})^{\otimes k}$. It follows that $\mathrm{H}^i(X, \mathrm{Sym}^k(E^{\oplus r})) = 0$ for all $r, k \geq 1$ and thus $\mathrm{H}^i(X, \mathrm{Sym}(E^{\oplus r})) = 0$ for all $r \geq 1$. $\square$

In the same way we obtain

Lemma 4.3. *Suppose that E is a B-module. Then*

$$\mathrm{H}^i(X, \wedge^r(E^{\oplus k})) = 0 \text{ for all } r, k \geq 1, i \geq r+1 \Longleftrightarrow \mathrm{H}^i(X, E^{\otimes r}) = 0 \text{ for all } r \geq 1, i \geq r+1.$$

Let us now consider the vanishing statement:

(4.4) $$\mathrm{H}^i(X, \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^{\oplus r})) = 0 \text{ for } i \geq 1.$$

Consider the short exact sequence

$$0 \to \mathfrak{b}^{\oplus r} \to \mathfrak{g}^{\oplus r} \to (\mathfrak{g}/\mathfrak{b})^{\oplus r} \to 0.$$

Its associated Koszul complex is

$$\cdots \wedge^{m-i}(\mathfrak{b}^{\oplus r}) \otimes \mathrm{Sym}^{m-i}(\mathfrak{g}^{\oplus r}) \to \cdots \to \mathrm{Sym}^m(\mathfrak{g}^{\oplus r}) \to \mathrm{Sym}^m((\mathfrak{g}/\mathfrak{b})^{\oplus r}) \to 0.$$

As the $\mathrm{Sym}^{m-i}(\mathfrak{g}^{\oplus r})$ are G-representations the corresponding G-equivariant vector bundles are trivial. We conclude that the vanishing statement (4.4) is equivalent to

(4.5) $$\mathrm{H}^{i+j}(X, \wedge^j(\mathfrak{b}^{\oplus r})) = 0 \text{ for all } i \geq 1.$$

This statement holds for any particular r. On the other hand, using Lemma 4.2, we see that the vanishing statement (4.4) for all r is equivalent to

$$\mathrm{H}^i(X, (\mathfrak{g}/\mathfrak{b})^{\otimes r}) = 0 \text{ for all } i, r \geq 1.$$

Similarly, using Lemma 4.3 we see that the vanishing statement (4.5) for all r is equivalent to

$$\mathrm{H}^i(X, \mathfrak{b}^{\otimes r}) = 0 \text{ for all } r \geq 1, \ i \geq r+1.$$

The short exact sequence

$$0 \to \mathfrak{b} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{b} \to 0$$

will give us further information. If we consider it as a two-step complex with cohomology $\mathfrak{g}/\mathfrak{b}$ and pass to the associated tensor complexes, we obtain a spectral sequence

(4.6) $$E_2^{p,-q} = \bigoplus_{\binom{r}{q}} \mathfrak{g}^{\otimes r-q} \otimes \mathrm{H}^p(X, \mathfrak{b}^{\otimes q}) \implies \mathrm{H}^{p-q}(X, (\mathfrak{g}/\mathfrak{b})^{\otimes r}).$$

From this we see immediately that

(4.7) $$H^i(X, \mathfrak{b}^{\otimes r}) = 0 \text{ for all } i \geq r.$$

implies, that

$$\mathfrak{g}^{\otimes r} \to H^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes r}) \qquad \text{is a surjection} .$$

Combining this with the previous discussion we obtain

Lemma 4.8. *The condition* (4.7) *implies that $\mathcal{A}_r$ has rational singularities for all values r.*

5. Cohomology of the vector bundles $\mathfrak{b}^{\otimes r}$ for type A_n

In the previous section we reduced the question of normality and the question of rational singularities of $\mathcal{A}_r$ to the study of the cohomology of $\mathfrak{b}^{\otimes r}$. We will now calculate the cohomology of these vector bundles for small r in type A_{n-1}, i.e., for $\mathfrak{g} = \mathfrak{sl}_n$. We will go sufficiently far to obtain our counterexamples, but will not make an attempt for a complete general answer.

5.1. **The case $r = 1$ and $\mathfrak{g}$ of any type.** This case is of course well known and easy and applies to any $\mathfrak{g}$, but we include the details in any case.

We will show that:

(5.1) $$H^k(X, \mathfrak{b}) = 0 \text{ for all } k.$$

Consider the short exact sequence

$$0 \to \mathfrak{n} \to \mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \to 0.$$

Since $\mathfrak{b}/\mathfrak{n}$ is a trivial vector bundle, $H^k(X, \mathfrak{b}/\mathfrak{n}) = 0$ for all $k \geq 1$. As by (3.2) we have $H^k(X, \mathfrak{n}) = 0$ for $k \neq 1$, it follows that

$$H^k(X, \mathfrak{b}) = 0 \text{ for } k \geq 2.$$

From the short exact sequence

$$0 \to \mathfrak{b} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{b} \to 0$$

we obtain the exact sequence

$$0 \to H^0(X, \mathfrak{b}) \to H^0(X, \mathfrak{g}) \to H^0(X, \mathfrak{g}/\mathfrak{b}) \to H^1(X, \mathfrak{b}) \to 0.$$

Moreover, $H^0(X, \mathfrak{g}) \cong \mathfrak{g}$ and the map $\mathfrak{g} \to H^0(X, \mathfrak{g}/\mathfrak{b})$ is an isomorphism as can be seen by a direct verification, for example. Thus we obtain (5.1).

5.2. The case $r = 2$ and $\mathfrak{g}$ of type A_{n-1}. Let $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 3$. We will show that

$$(5.2) \qquad \mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{b}) = \begin{cases} 0 & \text{if } k \neq 1 \\ \mathbb{C} & \text{if } k = 1. \end{cases}$$

and

$$(5.3) \qquad \mathrm{H}^k(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \cong \begin{cases} (\mathfrak{g} \otimes \mathfrak{g})/\mathbb{C} & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

First, we claim that

$$(5.4) \qquad \text{the potential support of } \mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{b}) \text{ is } \{0\} \text{ for all } k \geq 0.$$

Let $\chi \in \chi(\mathfrak{b} \otimes \mathfrak{b})$. We will show that

$$\text{either } \mathrm{H}^k(X, \mathcal{L}_\chi) = 0 \text{ for all } k \geq 0 \text{ or } \chi = w \cdot 0 \text{ for some } w \in W.$$

Recall that the bundle associate to the torus $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ is trivial and thus, by making use of Lemma 3.3, it suffices to prove the result for $\chi \in \chi(\mathfrak{n} \otimes \mathfrak{n})$. The case when χ is a sum of distinct negative roots follows from (3.1) and Lemma 3.3. So it remains to consider the case when $\chi \in \chi(\mathfrak{n} \otimes \mathfrak{n})$ and χ is not a sum of distinct negative roots. In that case $\chi = -2\alpha$ for a positive root $\alpha \in \Phi^+$ and $\mathrm{ht}(\alpha) = 1$ (if $\mathrm{ht}(\alpha) > 1$ then χ is a sum of distinct negative roots). Let us denote the set of simple roots with respect to Φ^+ by $\Delta := \{\alpha_i, i = 1, \ldots, n-1\}$. Let $\alpha = \alpha_i$, $1 \leq i \leq n-1$. If $i \leq n-2$, then $\langle -2\alpha_i + \rho, (\alpha_i + \alpha_{i+1})^\vee \rangle = 0$; if $i \geq 2$, then $\langle -2\alpha_i + \rho, (\alpha_{i-1} + \alpha_i)^\vee \rangle = 0$; thus $\mathrm{H}^k(X, \mathcal{L}_\chi) = 0$ for all k. This finishes the proof of (5.4).

Using the short exact sequence

$$0 \to \mathfrak{b} \otimes \mathfrak{b} \to \mathfrak{g} \otimes \mathfrak{b} \to \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b} \to 0.$$

and $\mathrm{H}^k(X, \mathfrak{g} \otimes \mathfrak{b}) \cong \mathfrak{g} \otimes \mathrm{H}^k(X, \mathfrak{b}) = 0$ for all k (we make use of (5.1)), we see that

$$\mathrm{H}^0(X, \mathfrak{b} \otimes \mathfrak{b}) = 0 \text{ and } \mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{b}) \cong \mathrm{H}^{k-1}(X, \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \text{ as } G\text{-modules for } k \geq 1.$$

Using the short exact sequence

$$0 \to \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to \mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b} \to (\mathfrak{g}/\mathfrak{b})^{\otimes 2} \to 0$$

and $\mathrm{H}^k(X, \mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b}) \cong \mathfrak{g} \otimes \mathrm{H}^k(X, \mathfrak{g}/\mathfrak{b}) = 0$ for $k \geq 1$, we see that

$$\mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \cong \mathrm{H}^{k-1}(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \text{ as } G\text{-modules for } k \geq 2$$

and that we have an exact sequence of G-modules

$$0 \to \mathrm{H}^0(X, \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \to \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \mathrm{H}^1(X, \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \to 0.$$

It follows that we have an isomorphism of G-modules

$$(5.5) \qquad \mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{b}) \cong \mathrm{H}^{k-2}(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \text{ for } k \geq 3$$

and an exact sequence of G-modules

$$(5.6) \qquad 0 \to \mathrm{H}^1(X, \mathfrak{b} \otimes \mathfrak{b}) \to \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \mathrm{H}^2(X, \mathfrak{b} \otimes \mathfrak{b}) \to 0.$$

As 0 is clearly not in the potential support of $\mathrm{H}^k(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2})$, it follows from (5.4), (5.5) and (5.6) that $\mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{b}) = 0$ for $k \geq 2$, and

$$\mathrm{H}^1(X, \mathfrak{b} \otimes \mathfrak{b}) \cong \mathrm{Hom}_G(\mathbb{C}, \mathfrak{g} \otimes \mathfrak{g}) \cong \mathrm{Hom}_G(\mathfrak{g}, \mathfrak{g}) \cong \mathbb{C}.$$

This completes the proof of (5.2). It also follows that

$$(5.7) \qquad \mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \cong \begin{cases} \mathbb{C} & \text{if } k = 0 \\ 0 & \text{if } k \geq 1 \end{cases}$$

Finally, we conclude (5.3) from (5.2), (5.5), and (5.6).

5.3. The case $r = 3$ and $\mathfrak{g}$ of type A_{n-1}.

Let $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 3$. We will show that

$$\mathrm{H}^k(X, \mathfrak{b}^{\otimes 3}) = 0, \ k \leq 1 \text{ or } k \geq 4;$$

$$(5.8) \qquad \mathrm{H}^2(X, \mathfrak{b}^{\otimes 3}) = \begin{cases} \mathbb{C}^{\oplus 2} \oplus L(\alpha_1 + \alpha_2)^{\oplus 5} \oplus L(2\alpha_1 + \alpha_2) \oplus L(\alpha_1 + 2\alpha_2) & \text{if } n = 3 \\ \mathbb{C}^{\oplus 2} & \text{if } n \geq 4; \end{cases}$$

$$\mathrm{H}^3(X, \mathfrak{b}^{\otimes 3}) = \begin{cases} L(\alpha_1 + 2\alpha_2 + \alpha_3) & \text{if } n = 4 \\ 0 & \text{if } n \neq 4. \end{cases}$$

We first show that

$$\text{if } n \geq 5 \quad \mathrm{PSupp}(\mathrm{H}^k(X, \mathfrak{b}^{\otimes 3})) = \{0\} \text{ for all } k,$$

$$\text{if } n = 4 \quad \mathrm{PSupp}(\mathrm{H}^k(X, \mathfrak{b}^{\otimes 3})) = \begin{cases} \{0, \alpha_1 + 2\alpha_2 + \alpha_3\} & \text{if } k = 3 \\ \{0\} & \text{if } k \neq 3, \end{cases}$$

$$(5.9)$$

$$\text{if } n = 3 \quad \mathrm{PSupp}(\mathrm{H}^k(X, \mathfrak{b}^{\otimes 3})) = \begin{cases} \{0, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\} & \text{if } k = 2 \\ \{0, \alpha_1 + \alpha_2\} & \text{if } k = 3 \\ \{0\} & \text{if } k \neq 2, 3. \end{cases}$$

Let $\chi \in \chi(\mathfrak{b}^{\otimes 3})$. Recall that the bundle associate to the torus $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ is trivial and thus, by making use of (5.4), it suffices to prove the result for $\chi \in \chi(\mathfrak{n}^{\otimes 3})$. If $\chi \in \chi(\wedge^3 \mathfrak{n})$, we see by (3.1) and Lemma 3.3 that either $H^k(X, \mathcal{L}_\chi) = 0$ for all k or $\chi = w \cdot 0$ for some $w \in W$. It remains to consider χ of the form $-2\alpha - \beta$, $\alpha, \beta \in \Phi^+$.

Let us introduce some notation. For $w \in W$, denote

$$\Phi_w := \{\gamma \in \Phi^+ \mid w^{-1}\gamma \in \Phi^-\}.$$

Suppose that $w = s_{i_1} s_{i_2} \cdots s_{i_l}$, where $s_i = s_{\alpha_i}$, is reduced expression. Recall that we have

$$\Phi_w = \{\alpha_{i_1}, \ s_{i_1}(\alpha_{i_2}), \ s_{i_1} s_{i_2}(\alpha_{i_3}), \ \cdots, \ s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l})\}.$$

In particular, the cardinality of Φ_w is the length $\ell(w)$.

For $\lambda \in X(T)$, we have

$$(5.10) \qquad w \cdot \lambda = w(\lambda) - \sum_{\gamma \in \Phi_w} \gamma.$$

Suppose first that $\alpha \neq \beta$. We show that if there exists $w \in W$ such that $w \cdot (-2\alpha - \beta) \in X^+(T)$, then $w \cdot (-2\alpha - \beta) = 0$ unless $n = 3$ and $\chi = -3\alpha_1 - \alpha_2$ or $-\alpha_1 - 3\alpha_2$.

As the dominant Weyl chamber is contained in the positive root cone, i.e., the inverse of the Cartan matrix has positive entries, we have

(5.11) $\qquad$ Let $\lambda \in \mathbb{Z}\Phi$ be an element in the root lattice which is dominant and not zero then $\lambda \in \mathbb{Z}_{>0}\alpha_1 + \cdots + \mathbb{Z}_{>0}\alpha_{n-1}$.

Assume that $w \cdot (-2\alpha - \beta)$ is dominant and not equal to zero. Making use of (5.10) and (5.11) we see that

$$w \cdot (-2\alpha - \beta) = -2w\alpha - w\beta - \sum_{\gamma \in \Phi_w} \gamma \in \mathbb{Z}_{>0}\alpha_1 + \cdots + \mathbb{Z}_{>0}\alpha_{n-1}.$$

Let us write $\alpha_0 = \alpha_1 + \ldots \alpha_{n-1}$ for the highest root and then we can rephrase the above equality as

$$-2w\alpha - w\beta - \sum_{\gamma \in \Phi_w} \gamma \geq \alpha_0.$$

Clearly at least one of $w\alpha$ or $w\beta$ has to be negative. Note that if $w\beta$ is negative then $-w\beta \in \Phi_w$ and similarly for $w\alpha$. Thus, if $w\alpha$ is negative and $w\beta$ is not we get

$$-w\alpha - w\beta - \sum_{\gamma \in \Phi_w - \{-w\alpha\}} \gamma \geq \alpha_0.$$

But this is impossible as $-w\alpha \leq \alpha_0$. Similarly we see that it is impossible for $w\alpha$ to be positive and for $w\beta$ to be negative. Hence, both $w\alpha$ and $w\beta$ have to be negative. In this case we see that

$$-w\alpha - \sum_{\gamma \in \Phi_w - \{-w\alpha, -w\beta\}} \gamma \geq \alpha_0.$$

This is only possble if $w\alpha = -\alpha_0$, $\Phi_w = \{\alpha_0, -w\beta\}$, and then also $\ell(w) = 2$. One sees directly that $w\alpha_0$ can be negative for $\ell(w) = 2$ only when $n = 3$. In that case the only possibilities are:

$$-2\alpha_1 - (\alpha_1 + \alpha_2) = (s_1 s_2) \cdot (\alpha_1 + \alpha_2), \quad -2\alpha_2 - (\alpha_1 + \alpha_2) = (s_2 s_1) \cdot (\alpha_1 + \alpha_2).$$

Thus we conclude that in this case the only possibility for potential support, in addition to 0, is $\alpha_1 + \alpha_2 \in \mathrm{PSupp}\mathrm{H}^2(X, \mathfrak{b}^{\otimes 3})$ when $n = 3$.

Suppose now that $\alpha = \beta$ and thus $\chi = -3\alpha$. If $\mathrm{ht}\,\alpha \geq 3$, then χ is a sum of distinct negative roots and we see by (3.1) and Lemma 3.3 that either $H^k(X, \mathcal{L}_\chi) = 0$ for all k or $\chi = w \cdot 0$ for some $w \in W$. Let us assume next that $\mathrm{ht}\,\alpha = 2$ and we write $\alpha = \alpha_i + \alpha_{i+1}$, $1 \leq i \leq n-2$. If $i \leq n-3$, then $\langle -3\alpha + \rho, (\alpha_i + \alpha_{i+1} + \alpha_{i+2})^\vee \rangle = 0$ and if $i \geq 2$, then $\langle -3\alpha + \rho, (\alpha_{i-1} + \alpha_i + \alpha_{i+1})^\vee \rangle = 0$. Thus in these cases $\mathrm{H}^k(X, \mathcal{L}_\chi) = 0$ for all k and they do not contribute to the potential support. Therefore we are left to consider the case when $n = 3$ and $i = 1$. Then

$$-3(\alpha_1 + \alpha_2) = (s_1 s_2 s_1) \cdot (\alpha_1 + \alpha_2).$$

Hence $\alpha_1 + \alpha_2 \in \mathrm{PSupp}\mathrm{H}^3(X, \mathfrak{b}^{\otimes 3})$ when $n = 3$.

Finally, let us assume that $\operatorname{ht}\alpha = 1$ and so $\alpha = \alpha_i$, $1 \leq i \leq n - 1$. If $i \leq n - 3$, we have $\langle -3\alpha + \rho, (\alpha_i + \alpha_{i+1} + \alpha_{i+2})^\vee \rangle = 0$ and if $i \geq 3$, we have $\langle -3\alpha + \rho, (\alpha_{i-2} + \alpha_{i-1} + \alpha_i)^\vee \rangle = 0$. This shows that $\operatorname{H}^k(X, \mathcal{L}_\chi) = 0$ for all k. Thus we get no contribution to the potential support in these cases. This leaves us with the possibilities of $n = 3$ with $i = 1, 2$ and $n = 4$ with $i = 2$. In these cases, we have

$$-3\alpha_1 = (s_1 s_2) \cdot (2\alpha_1 + \alpha_2), \quad -3\alpha_2 = (s_2 s_1) \cdot (\alpha_1 + 2\alpha_2) \quad (n = 3)$$

$$-3\alpha_2 = (s_2 s_1 s_3) \cdot (\alpha_1 + 2\alpha_2 + \alpha_3) \quad (n = 4).$$

Thus we obtain $2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 \in \operatorname{PSuppH}^2(X, \mathfrak{b}^{\otimes 3})$ when $n = 3$ and $\alpha_1 + 2\alpha_2 + \alpha_3 \in \operatorname{PSuppH}^3(X, \mathfrak{b}^{\otimes 3})$ when $n = 4$.

This completes the proof of (5.9) and we now turn to the proof of (5.8).

Using the short exact sequence

$$0 \to \mathfrak{b}^{\otimes 3} \to \mathfrak{g} \otimes \mathfrak{b}^{\otimes 2} \to \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b} \to 0$$

and (5.2) we see that we have

$$\operatorname{H}^0(X, \mathfrak{b}^{\otimes 3}) = 0, \ \operatorname{H}^k(X, \mathfrak{b}^{\otimes 3}) \cong \operatorname{H}^{k-1}(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \text{ for } k \geq 3,$$

and also an exact sequence

$$(5.12) \quad 0 \to \operatorname{H}^0(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \to \operatorname{H}^1(X, \mathfrak{b}^{\otimes 3}) \to \mathfrak{g} \to \operatorname{H}^1(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \to \operatorname{H}^2(X, \mathfrak{b}^{\otimes 3}) \to 0.$$

Using the short exact sequence

$$0 \to \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b} \to \mathfrak{g} \otimes \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2} \to 0$$

and (5.7) we obtain

$$\operatorname{H}^k(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \cong \operatorname{H}^{k-1}(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \text{ for } k \geq 2$$

and an exact sequence

$$(5.13) \qquad 0 \to \operatorname{H}^0(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \to \mathfrak{g} \to \operatorname{H}^0(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \operatorname{H}^1(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \to 0.$$

Using the short exact sequence

$$0 \to \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2} \to \mathfrak{g} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2} \to (\mathfrak{g}/\mathfrak{b})^{\otimes 3} \to 0$$

and (5.3) we obtain

$$\operatorname{H}^k(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \cong \operatorname{H}^{k-1}(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \text{ for } k \geq 2$$

and an exact sequence

$$0 \to \operatorname{H}^0(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \mathfrak{g} \otimes \operatorname{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \operatorname{H}^0(X, \mathfrak{g}/\mathfrak{b}^{\otimes 3}) \to \operatorname{H}^1(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to 0.$$

It follows that we have an isomorphism of G-modules

$$\operatorname{H}^k(X, \mathfrak{b}^{\otimes 3}) \cong \operatorname{H}^{k-3}(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \text{ for } k \geq 4,$$

and an exact sequence of G-modules

$$(5.14) \ 0 \to \operatorname{H}^0(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \mathfrak{g} \otimes \operatorname{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \operatorname{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \to \operatorname{H}^3(X, \mathfrak{b}^{\otimes 3}) \to 0.$$

Thus, as 0 is not in the potential support of $H^k(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3})$, we conclude that

(5.15) $H^k(X, \mathfrak{b}^{\otimes 3})$ does not contain the trivial representation for $k \geq 3$.

Thus, we conclude from (5.9) that $H^k(X, \mathfrak{b}^{\otimes 3}) = 0$ for $k \geq 4$.

Also, it follows from (5.9), (5.12), and (5.13) that

(5.16) $H^0(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \cong H^1(X, \mathfrak{b}^{\otimes 3}) = 0.$

Thus, we have shown, in particular, that $H^k(X, \mathfrak{b}^{\otimes 3}) = 0$ for $k = 0, 1$. Thus we have obtained the first claim of (5.8). Before proceeding further, we record one more general fact which we obtain from (5.16), (5.12), and (5.13):

(5.17a) $H^0(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \cong H^1(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \oplus \mathfrak{g} \cong H^2(X, \mathfrak{b}^{\otimes 3}) \oplus \mathfrak{g}^{\oplus 2}$

(5.17b) $H^1(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \cong H^2(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) \cong H^3(X, \mathfrak{b}^{\otimes 3})$

(5.17c) $H^{k-1}(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \cong H^k(X, \mathfrak{b}^{\otimes 2} \otimes \mathfrak{g}/\mathfrak{b}) = 0$ if $k \neq 1, 2$.

We now argue with specific values of n.

Assume that $n \geq 5$. It follows from (5.15) and (5.9) that

$$H^3(X, \mathfrak{b}^{\otimes 3}) = 0 \quad \text{and} \quad H^2(X, \mathfrak{b}^{\otimes 3}) \cong \mathbb{C}^{\oplus c},$$

for some c which we determine later.

Assume that $n = 4$. It follows from (5.15) and (5.9) that

$$H^3(X, \mathfrak{b}^{\otimes 3}) \cong L(\alpha_1 + 2\alpha_2 + \alpha_3) \quad \text{and} \quad H^2(X, \mathfrak{b}^{\otimes 3}) \cong \mathbb{C}^{\oplus a},$$

for some a which we determine later.

Assume that $n = 3$. As $\alpha_1 + \alpha_2$ is not in the potential support of $H^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3})$ we conclude from (5.9), (5.14) and (5.15) that

$$H^3(X, \mathfrak{b}^{\otimes 3}) = 0.$$

Thus $H^2(X, \mathfrak{b}^{\otimes 3})$ is the only non-vanishing cohomology group in this case. Making use of Borel-Weil-Bott (3.1) just as in our argument for (5.9) we see that

$$H^*(X, \mathrm{gr}(\mathfrak{b}^{\otimes 3})) \cong \mathbb{C}^{\oplus b} + L(\alpha_1 + \alpha_2)^{\oplus 5} + L(2\alpha_1 + \alpha_2) + L(\alpha_1 + 2\alpha_2).$$

for some b. Therefore we get:

$$H^2(X, \mathfrak{b}^{\otimes 3}) \cong H^*(X, \mathrm{gr}(\mathfrak{b}^{\otimes 3})) \cong \mathbb{C}^{\oplus b} \oplus L(\alpha_1 + \alpha_2)^{\oplus 5} \oplus L(2\alpha_1 + \alpha_2) \oplus L(\alpha_1 + 2\alpha_2).$$

We now determine a, b, and c. Now,

$$\mathbb{C}^{\oplus a} \cong \mathbb{C}^{\oplus b} \cong \mathbb{C}^{\oplus c} \cong \mathrm{Hom}_G(\mathbb{C}, H^2(X, \mathfrak{b}^{\otimes 3})).$$

By (5.17a) we get

$$\mathrm{Hom}_G(\mathbb{C}, H^2(X, \mathfrak{b}^{\otimes 3})) \cong \mathrm{Hom}_G(\mathbb{C}, H^2(X, \mathfrak{b}^{\otimes 3}) \oplus \mathfrak{g}^{\oplus 2}) \cong \mathrm{Hom}_G(\mathbb{C}, H^0(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2})) \cong$$

$$\cong \mathrm{Hom}_G(\mathbb{C}, \mathfrak{g} \otimes H^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2})) \cong (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})^G \cong \mathbb{C}^{\oplus 2};$$

where in the third equality we have made use of (5.14) and the fact that the trivial representation does not occur in $\mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3})$, in the fourth equality we made use of (5.3). The last equality is classical and can also be verified by a direct calculation: the two invariant tensors are $(x, y, z) \mapsto \mathrm{Tr}(xyz)$ and $(x, y, z) \mapsto \mathrm{Tr}(yxz)$. This completes the proof of (5.8).

5.4. The case $r = 4$ and $\mathfrak{g}$ of type A_{n-1}. Let $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 6$.

In the previous cases we obtained complete information of the cohomology for all values of $n \geq 3$. For $r = 4$ we will not make an attempt to get a complete answer, but will just obtain enough information for our counterexample. In particular, we already have enough information to prove the Cohen-Macauley property for $\mathfrak{sl}_3$. In the cases $n = 4, 5$ the answer is probably obtainable with our techniques but is more complicated.

We will show that if $n \geq 7$, then

$$(5.18a) \qquad\qquad \mathrm{H}^k(X, \mathfrak{b}^{\otimes 4}) = 0 \text{ if } k \neq 2, 3$$

and if $n = 6$, then

$$(5.18b) \qquad \mathrm{H}^k(X, \mathfrak{b}^{\otimes 4}) = \begin{cases} L(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5) & \text{if } k = 5 \\ 0 & \text{if } k \neq 2, 3, 5 \,. \end{cases}$$

Remark 5.19. *The $\mathrm{H}^2(X, \mathfrak{b}^{\otimes 4})$ and $\mathrm{H}^3(X, \mathfrak{b}^{\otimes 4})$ both consist of a number of copies of the trivial representation.*

We first show that

$$(5.20a) \qquad\qquad \text{if } n \geq 7 \quad \mathrm{PSuppH}^k(X, \mathfrak{b}^{\otimes 4}) = \{0\} \text{ for all } k \geq 0;$$

and

$$(5.20b) \qquad \text{if } n = 6 \ \ \mathrm{PSuppH}^k(X, \mathfrak{b}^{\otimes 4}) = \begin{cases} \{0\} & \text{if } k \neq 5 \\ \{0, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5\} & \text{if } k = 5 \,. \end{cases}$$

Let $\chi \in \chi(\mathfrak{b}^{\otimes 4})$. Recall that the bundle associate to the torus $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ is trivial. Thus, by making use of (5.9), it suffices to prove the result for $\chi \in \chi(\mathfrak{n}^{\otimes 4})$. If $\chi \in \chi(\wedge^4 \mathfrak{n})$, we see by (3.1) and Lemma 3.3 that either $H^k(X, \mathcal{L}_\chi) = 0$ for all k or $\chi = w \cdot 0$ for some $w \in W$. Thus we are reduced to considering the case when χ is not a sum of distinct roots.

Assume first that $\chi = -2\alpha - \beta - \gamma$, where $\alpha, \beta, \gamma \in \Phi^+$ are distinct. We will make use of the notation and argue in the similar manner as in the previous subsection 5.3 . Suppose that there exists a $w \in W$, such that $w \cdot \chi \in X^+(T)$ and $w \cdot \chi \neq 0$. Making use of (5.10) and (5.11) we conclude, as in subsection 5.3, that

$$w \cdot (-2\alpha - \beta - \gamma) = w(-2\alpha - \beta - \gamma) - \sum_{\gamma \in \Phi_w} \gamma \geq \alpha_0 \,.$$

Further arguing as in subsection 5.3 we conclude that w has to satisfy that $w\alpha = -\alpha_0$, $w\beta < 0$, $w\gamma < 0$ and $\ell(w) = 3$. But, $\alpha_0 \in \Phi_w$ only if $l(w) \geq n-1$ and we have assumed that $n \geq 6$.

Assume next that $\chi = -2\alpha - 2\beta$, where $\alpha, \beta \in \Phi^+$ are distinct. Suppose that there exists a $w \in W$, such that $w \cdot \chi \in X^+(T)$ and $w \cdot \chi \neq 0$. Making use of (5.11) again we see that

$$w \cdot (-2\alpha - 2\beta + \rho) - \rho = \alpha_0 + \lambda_0$$

for some $\lambda_0 \in \mathbb{Z}_+ \Phi^+$. In particular, we have

$$\langle -2\alpha - 2\beta + \rho, -2\alpha - 2\beta + \rho \rangle = \langle w(-2\alpha - 2\beta + \rho), w(-2\alpha - 2\beta + \rho) \rangle$$
$$= \langle \alpha_0 + \lambda + \rho, \alpha_0 + \lambda + \rho \rangle \geq \langle \alpha_0 + \rho, \alpha_0 + \rho \rangle = 2n + \langle \rho, \rho \rangle.$$

It follows that

$$\langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle + 2\langle \alpha, \beta \rangle - \langle \alpha + \beta, \rho \rangle \geq n/2.$$

This can only happen when

$$n = 6, \ \langle \alpha, \beta \rangle = 1, \langle \alpha, \rho \rangle = 2, \ \langle \beta, \rho \rangle = 1.$$

Here we can and assume that $\mathrm{ht}\alpha = \langle \alpha, \rho \rangle \geq \mathrm{ht}\beta = \langle \beta, \rho \rangle$.

Suppose that $n = 6$ and $\beta = \alpha_i$, $1 \leq i \leq 5$. Then $\alpha = \alpha_i + \alpha_{i+1}$ or $\alpha = \alpha_{i-1} + \alpha_i$. In the first case we have that $-2\alpha - 2\beta = -4\alpha_i - 2\alpha_{i+1}$, and

$$\begin{cases} \langle -4\alpha_i - 2\alpha_{i+1} + \rho, (\alpha_{i-1} + \alpha_i)^\vee \rangle = 0 & \text{if } i \geq 2 \\ \langle -4\alpha_i - 2\alpha_{i+1} + \rho, (\sum_{j=i}^{i+3} \alpha_j)^\vee \rangle = 0 & \text{if } i \leq 2. \end{cases}$$

In the second case we have that $-2\alpha - 2\beta = -2\alpha_{i-1} - 4\alpha_i$ and

$$\begin{cases} \langle -2\alpha_{i-1} - 4\alpha_i + \rho, (\alpha_i + \alpha_{i+1})^\vee \rangle = 0 & \text{if } i \leq 4 \\ \langle -2\alpha_{i-1} - 4\alpha_i + \rho, (\sum_{j=i-3}^{i} \alpha_j)^\vee \rangle = 0 & \text{if } i \geq 4. \end{cases}$$

It follows that in these cases $-2\alpha - 2\beta$ does not contribute to the potential support as $H^k(X, \mathcal{L}_{-2\alpha-2\beta}) = 0$ for all k.

Assume that $\chi = -3\alpha - \beta$, where $\alpha, \beta \in \Phi^+$ are distinct roots. Suppose that there exists a $w \in W$, such that $w \cdot \chi \in X^+(T)$ and $w \cdot \chi \neq 0$. Argue as above, we have

$$9\langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle + 6\langle \alpha, \beta \rangle - 6\langle \alpha, \rho \rangle - 2\langle \beta, \rho \rangle \geq 2n.$$

This can happen only if $n \leq 8$, $\langle \alpha, \beta \rangle = 1$, $3\langle \alpha, \rho \rangle + \langle \beta, \rho \rangle \leq 13 - n$, or if $n = 6$, $\langle \alpha, \beta \rangle = 0$, and $\langle \alpha, \rho \rangle = \langle \beta, \rho \rangle = 1$. More precisely, if $n = 6$ and $\langle \alpha, \beta \rangle = 1$, then the possible values for $(\langle \alpha, \rho \rangle, \langle \beta, \rho \rangle)$ are $(1, 2), (1, 3), (1, 4), (2, 1)$; if $n = 7$ and $\langle \alpha, \beta \rangle = 1$, then the possible values for $(\langle \alpha, \rho \rangle, \langle \beta, \rho \rangle)$ are $(1, 2), (1, 3)$; and if $n = 8$ and $\langle \alpha, \beta \rangle = 1$, then $(\langle \alpha, \rho \rangle, \langle \beta, \rho \rangle) = (1, 2)$.

Suppose first that $\langle \alpha, \beta \rangle = 1$ and $\alpha = \alpha_i$, $1 \leq i \leq n - 1$. Then $\beta = \sum_{k=i}^{j} \alpha_k$ for some $j \geq i+1$ or $\beta = \sum_{k=j}^{i} \alpha_k$ for some $j \leq i-1$. In the first case we have $-3\alpha - \beta = -4\alpha_i - \sum_{k=i+1}^{j} \alpha_k$ and

$$\begin{aligned} \langle -3\alpha - \beta + \rho, \alpha_j^\vee \rangle = 0 & \qquad \text{if } j \neq i+1 \\ \langle -3\alpha - \beta + \rho, (\sum_{k=i}^{i+3} \alpha_k)^\vee \rangle = 0 & \qquad \text{if } j = i+1 \text{ and } i \leq n-4 \\ \langle -3\alpha - \beta + \rho, (\sum_{k=i-2}^{i} \alpha_k)^\vee \rangle = 0 & \qquad \text{if } j = i+1 \text{ and } i \geq 3; \end{aligned}$$

in the second case we have $-3\alpha - \beta = -\sum_{k=j}^{i-1} \alpha_k - 4\alpha_i$ and

$$\begin{aligned}
\langle -3\alpha - \beta + \rho, \alpha_j^\vee \rangle &= 0 & \text{if } j \neq i - 1 \\
\langle -3\alpha - \beta + \rho, (\textstyle\sum_{k=i}^{i+2} \alpha_k)^\vee \rangle &= 0 & \text{if } j = i - 1 \text{ and } i \leq n - 3 \\
\langle -3\alpha - \beta + \rho, (\textstyle\sum_{k=i-3}^{i} \alpha_k)^\vee \rangle &= 0 & \text{if } j = i - 1 \text{ and } i \geq 4.
\end{aligned}$$

Thus in all these cases $-3\alpha - \beta$ dose not contribute to the potential support.

Suppose next that $n = 6$, $\langle \alpha, \beta \rangle = 1$, $\langle \alpha, \rho \rangle = 2$ and $\langle \beta, \rho \rangle = 1$. Let us write $\beta = \alpha_i$. Then $\alpha = \alpha_i + \alpha_{i+1}$ or $\alpha = \alpha_{i-1} + \alpha_i$. We have

$$\begin{aligned}
\langle -4\alpha_i - 3\alpha_{i+1} + \rho, (\textstyle\sum_{k=i-1}^{i+1} \alpha_k)^\vee \rangle &= 0 & \text{if } i \geq 2 \\
\langle -4\alpha_i - 3\alpha_{i+1} + \rho, (\textstyle\sum_{k=i}^{i+3} \alpha_k)^\vee \rangle &= 0 & \text{if } i \leq 2
\end{aligned}$$

or

$$\begin{aligned}
\langle -3\alpha_{i-1} - 4\alpha_i + \rho, (\textstyle\sum_{k=i-1}^{i+1} \alpha_k)^\vee \rangle &= 0 & \text{if } i \leq 4 \\
\langle -3\alpha_{i-1} - 4\alpha_i + \rho, (\textstyle\sum_{k=i-3}^{i} \alpha_k)^\vee \rangle &= 0 & \text{if } i \geq 4.
\end{aligned}$$

Thus again in this case $-3\alpha - \beta$ dose not contribute to the potential support.

Suppose now that $n = 6$, $\langle \alpha, \beta \rangle = 0$, $\langle \alpha, \rho \rangle = \langle \beta, \rho \rangle = 1$. Let us write $\alpha = \alpha_i$ and $\beta = \alpha_j$. Then either $j \geq i + 2$ or $j \leq i - 2$. We have

$$\begin{aligned}
\langle -3\alpha_i - \alpha_j + \rho, (\textstyle\sum_{k=i}^{i+2} \alpha_k)^\vee \rangle &= 0 & \text{if } j > i + 3 \\
\langle -3\alpha_i - \alpha_j + \rho, (\textstyle\sum_{k=i}^{i+1} \alpha_k)^\vee \rangle &= 0 & \text{if } j = i + 2 \\
\langle -3\alpha_i - \alpha_j + \rho, (\textstyle\sum_{k=i}^{i+3} \alpha_k)^\vee \rangle &= 0 & \text{if } j = i + 3 \\
\langle -3\alpha_i - \alpha_j + \rho, (\textstyle\sum_{k=i-2}^{i} \alpha_k)^\vee \rangle &= 0 & \text{if } j < i - 3 \\
\langle -3\alpha_i - \alpha_j + \rho, (\textstyle\sum_{k=i-1}^{i} \alpha_k)^\vee \rangle &= 0 & \text{if } j = i - 2 \\
\langle -3\alpha_i - \alpha_j + \rho, (\textstyle\sum_{k=i-3}^{i} \alpha_k)^\vee \rangle &= 0 & \text{if } j = i - 3.
\end{aligned}$$

Thus also in this case $-3\alpha - \beta$ dose not contribute to the potential support.

Finally, assume that $\chi = -4\alpha$. If $\operatorname{ht}(\alpha) \geq 4$, then χ is a sum of distinct negative roots, so we can assume that $\operatorname{ht}(\alpha) \leq 3$. Suppose that $\alpha = \alpha_i + \alpha_{i+1} + \alpha_{i+2}$. If $i \leq n - 4$, then $\langle -4\alpha + \rho, (\sum_{j=i}^{i+3} \alpha_j)^\vee \rangle = 0$; if $i \geq 2$, then $\langle -4\alpha + \rho, (\sum_{j=i-1}^{i+2} \alpha_j)^\vee \rangle = 0$; thus $\mathrm{H}^k(X, \mathcal{L}_\chi) = 0$ for all k.

Suppose that $\alpha = \alpha_i + \alpha_{i+1}$. If $i \leq n - 4$, then $\langle -4\alpha + \rho, (\sum_{j=i}^{i+3} \alpha_j)^\vee \rangle = 0$; if $i \geq 3$, then $\langle -4\alpha + \rho, (\sum_{j=i-2}^{i+1} \alpha_j)^\vee \rangle = 0$ and thus $\mathrm{H}^k(X, \mathcal{L}_\chi) = 0$ for all k.

Suppose that $\alpha = \alpha_i$. If $i \leq n - 4$, then $\langle -4\alpha + \rho, (\sum_{j=i}^{i+3} \alpha_j)^\vee \rangle = 0$; if $i \geq 4$, then $\langle -4\alpha + \rho, (\sum_{j=i-3}^{i} \alpha_j)^\vee \rangle = 0$; thus $\mathrm{H}^k(X, \mathcal{L}_\chi) = 0$ for all k unless when $n = 6$ and $i = 3$. In this case we have

$$-4\alpha_3 = (s_5 s_1 s_4 s_2 s_3) \cdot (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5).$$

This completes the proof of (5.20).

Consider the short exact sequences

$$0 \to \mathfrak{b} \otimes \mathfrak{b}^{\otimes i} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes(3-i)} \to \mathfrak{g} \otimes \mathfrak{b}^{\otimes i} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes(3-i)} \to \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b}^{\otimes i} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes(3-i)} \to 0, \ i = 0, 1, 2, 3.$$

From the exact sequence with $i = 3$ we conclude, making use of (5.8), that

$$\mathrm{H}^0(X, \mathfrak{b}^{\otimes 4}) = 0, \ \mathrm{H}^k(X, \mathfrak{b}^{\otimes 4}) \cong \mathrm{H}^{k-1}(X, \mathfrak{b}^{\otimes 3} \otimes \mathfrak{g}/\mathfrak{b}) \text{ for } k \geq 4 \text{ or } k = 1$$

and we further obtain the exact sequence

$$0 \to \mathrm{H}^1(X, \mathfrak{b}^{\otimes 3} \otimes \mathfrak{g}/\mathfrak{b}) \to \mathrm{H}^2(X, \mathfrak{b}^{\otimes 4}) \to \mathfrak{g}^{\oplus 2} \to \mathrm{H}^2(X, \mathfrak{b}^{\otimes 3} \otimes \mathfrak{g}/\mathfrak{b}) \to \mathrm{H}^3(X, \mathfrak{b}^{\otimes 4}) \to 0 \,.$$

Using the exact sequence with $i = 2$ and making use of (5.17) and (5.8) we get

$$\mathrm{H}^0(X, \mathfrak{b}^{\otimes 3} \otimes \mathfrak{g}/\mathfrak{b}) = 0, \ \mathrm{H}^k(X, \mathfrak{b}^{\otimes 3} \otimes \mathfrak{g}/\mathfrak{b}) \cong \mathrm{H}^{k-1}(X, \mathfrak{b}^{\otimes 2} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \text{ for } k \geq 3$$

and we obtain the exact sequence

$$0 \to \mathrm{H}^0(X, \mathfrak{b}^{\otimes 2} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \mathrm{H}^1(X, \mathfrak{b}^{\otimes 3} \otimes \mathfrak{g}/\mathfrak{b}) \to \mathfrak{g}^{\oplus 2} \oplus (\mathfrak{g} \otimes \mathfrak{g})^{\oplus 2} \to$$
$$\to \mathrm{H}^1(X, \mathfrak{b}^{\otimes 2} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \mathrm{H}^2(X, \mathfrak{b}^{\otimes 3} \otimes \mathfrak{g}/\mathfrak{b}) \to 0 \,.$$

Using the exact sequence with $i = 1$ and making use of (5.17) and (5.8) we get

$$\mathrm{H}^k(X, \mathfrak{b}^{\otimes 2} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \cong \mathrm{H}^{k-1}(X, \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}^{\otimes 3}) \text{ for } k \geq 2,$$

and the exact sequence

$$0 \to \mathrm{H}^0(X, \mathfrak{b}^{\otimes 2} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to \mathfrak{g}^{\oplus 2} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \to \mathrm{H}^0(X, \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}^{\otimes 3}) \to \mathrm{H}^1(X, \mathfrak{b}^{\otimes 2} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 2}) \to 0 \,.$$

Using the exact sequence with $i = 0$ and making use of (5.17) and (5.8) we get

$$\mathrm{H}^k(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \cong \mathrm{H}^{k-1}(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 4}), \ k \geq 2,$$

and we obtain the exact sequence

$$0 \to \mathrm{H}^0(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \to \mathfrak{g} \otimes \mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \to \mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 4}) \to \mathrm{H}^1(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \to 0.$$

It follows that

$$(5.21) \qquad \mathrm{H}^k(X, \mathfrak{b}^{\otimes 4}) = 0, \ k = 0, 1 \ \text{ and } \ \mathrm{H}^k(X, \mathfrak{b}^{\otimes 4}) \cong \mathrm{H}^{k-4}(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 4}), \ k \geq 5;$$

and

$$(5.22) \quad 0 \to \mathrm{H}^0(X, \mathfrak{b} \otimes (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \to \mathfrak{g} \otimes \mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3}) \to$$
$$\to \mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 4}) \to H^4(X, \mathfrak{b}^{\otimes 4}) \to 0.$$

As 0 is not in the potential support of $H^k(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 4})$, using (5.20), (5.21) and (5.22) we see that (5.18) follows.

5.5. **Type B_2.** As the case $r = 1$ applies to any type, we consider the case $r = 2$ here. We write $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$, where $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and $\langle \alpha_2, \alpha_1^\vee \rangle = -2$.

Just as in subsection 5.2 one checks that:
$$\mathrm{PSupp}(\mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{b})) = \begin{cases} \{0\} & \text{if } k = 0, 1, 3 \\ \{0, \alpha_1 + \alpha_2\} & \text{if } k = 2 \\ \emptyset & \text{if } k \geq 4. \end{cases}$$

Proceeding as in 5.2 we see that:

(5.23)
$$\mathrm{H}^k(X, \mathfrak{b} \otimes \mathfrak{b}) = \begin{cases} 0 & \text{if } k = 0 \text{ of } k \geq 3 \\ \mathbb{C} & \text{if } k = 1 \\ L(\alpha_1 + \alpha_2) & \text{if } k = 2 \,. \end{cases}$$

The only difference to the argument in subsection 5.2 is that the weight $\alpha_1 + \alpha_2$ occurs for $k = 2$ only and it occurs precisely once in that case; otherwise the argument is the same.

6. Results and counterexamples

6.1. **The case of A_2.** In this case we have the following:

Theorem 6.1. *For $\mathfrak{g} = \mathfrak{sl}_3$ the variety $\mathcal{A}_r$ is Cohen-Macauley for all r.*

Proof. According to lemma 4.8 it suffices to show that
$$\mathrm{H}^i(X, \mathfrak{b}^{\otimes r}) = 0 \text{ for all } i \geq r \geq 1.$$
By dimension reasons we only have to consider the cases $r = 1, 2, 3$. These cases are treated in subsections 5.1, 5.2, and 5.3, respectively.

$\square$

Remark 6.2. *By a slight modification of our methods one can show that this result holds for all characteristics above 3.*

6.2. **The case of B_2 and $r = 2$.** We will show that $\mathcal{A}_2$ is not normal in the case of B_2. For $\mathcal{A}_2$ to be normal, the map
$$\mathrm{Sym}(\mathfrak{g} \oplus \mathfrak{g}) \to \mathrm{H}^0(X, \mathrm{Sym}(\mathfrak{g}/\mathfrak{b} \oplus \mathfrak{g}/\mathfrak{b}))$$
has to be onto. In particular this has to hold for Sym^2 and hence, by decomposing the Sym^2 on both sides, the map
$$\mathfrak{g} \otimes \mathfrak{g} \to \mathrm{H}^0(X, \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b})$$
has to be onto. Making use of the spectral sequence (4.6), (5.1), and (5.23) we get that

$$E_2^{p,-q} = \bigoplus_{\binom{2}{q}} \mathfrak{g}^{\otimes 2 - q} \otimes \mathrm{H}^p(X, \mathfrak{b}^{\otimes q}) = \begin{cases} \mathfrak{g}^{\otimes 2} & \text{if } p = q = 0 \\ L(\alpha_1 + \alpha_2) & \text{if } p = q = 2 \\ 0 & \text{otherwise if } p - q \geq 0 \,. \end{cases}$$

As the term $E_2^{2,-2}$ must survive in the spectral sequence we see that $\mathfrak{g}^{\otimes 2} \to \mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 2})$ cannot be onto and so $\mathcal{A}_2$ is not normal in this case.

6.3. **The case of A_3 and $r = 3$.** Making use of the spectral sequence (4.6), (5.1), (5.2), and (5.8) we get that

$$E_2^{p,-q} = \bigoplus_{\binom{3}{q}} \mathfrak{g}^{\otimes 3-q} \otimes \mathrm{H}^p(X, \mathfrak{b}^{\otimes q}) \ = \ \begin{cases} \mathfrak{g}^{\otimes 3} & \text{if } p = q = 0 \\ L(\alpha_1 + 2\alpha_2 + \alpha_3) & \text{if } p = q = 3 \\ 0 & \text{otherwise if } p - q \geq 0 \,. \end{cases}$$

As the term $E_2^{3,-3}$ must survive in the spectral sequence we see that $\mathfrak{g}^{\otimes 3} \to \mathrm{H}^0(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 3})$ cannot be onto and so, arguing as in the B_2 case above, we see that $\mathcal{A}_3$ is not normal.

6.4. **The case of A_5 and $r = 4$.** Making use of the spectral sequence (4.6), (5.1), (5.2), (5.8), and (5.18) we see that

$$E_2^{p,-q} = \bigoplus_{\binom{3}{q}} \mathfrak{g}^{\otimes 3-q} \otimes \mathrm{H}^p(X, \mathfrak{b}^{\otimes q}) \ = \ \begin{cases} L(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5) & \text{if } p = 5 \text{ and } q = 4 \\ 0 & \text{otherwise if } p - q > 0 \,. \end{cases}$$

As the term $E_2^{5,-4}$ must survive in the spectral sequence we see that $\mathrm{H}^1(X, (\mathfrak{g}/\mathfrak{b})^{\otimes 4}) = L(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5)$. In particular, it is not zero and hence in this case the normalization of $\mathcal{A}_4$ does not have rational singularities.

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